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ON THE LOW REYNOLDS NUMBER MOTION OF TWO DROPLETS

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Abstract—Exact solutions are derived for the quasi steady-state creeping flow internal and external to two spherical droplets moving along their line-of-centers. Numerical results are presented, which include all previous solutions as special cases.

1. INTRODUCTION

The hydrodynamics of multiple droplets' motion in an unbounded medium is of fundamental importance and of much interest in numerous engineering applications. When the droplets are small and when their relative velocities are low, their motion can be described by the Stokes equations. Solutions for the motion of a single droplet suspended in an unbounded medium or in a tube have been presented previously for a variety of cases (Hetsroni *et al.* 1970a, b, 1971; Greenstein 1972). However, when the concentration of the droplets is of $O(10^{-2})$, the interaction between them becomes significant. Droplets may possibly collide, which makes collision courses of interest.

There exists substantial literature devoted to the computation of collision efficiencies. The most accepted theory of small cloud droplets' collision under electrically neutral conditions, is that of Hocking (1959). He considered the motion of two solid spherical particles in a quiescent field, and used Stokes equations to calculate the collision efficiencies versus the radii of the particles, with the radius of the larger particle as a parameter. Hocking stated incorrectly that no collision occurs between droplets smaller than 19 μm. This error was pointed out by Davis & Sartor (1967) and was subsequently corrected by Hocking & Jonas (1970) who showed that collision efficiencies are finite for all sizes of particles, but vary considerably with the gap between the particles. All these solutions are based on quite arbitrary concepts, such as Hocking's "gap space", since collisions between two particles or droplets are impossible to treat purely on the basis of the solutions of the Stokes equations.

Other studies are concerned with the exact solution of the Stokes equations. The only systems for which exact solutions have been obtained are the two-sphere system or the combination of a sphere and a plane. We shall find here that all these solutions are particular cases of the general solution which we present herein.

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Jeffery (1915) solved the motion of two rigid spheres which rotate slowly about their line of centers. Later Stimson & Jeffery (1926) solved the axisymmetric problem of two spheres translating at an equal velocity along their line of centers. Cooley & O'Neill (1969) recalculated the results of Stimson & Jeffery for the more general case when the two spheres are not of equal size. In other publications O'Neill and co-workers (1963, 1969, 1970) solved the asymmetrical motion of two equal spheres and the motion of two spheres approaching each other or a solid wall. These, together with the calculations of Goldman, Cox & Brenner (1967), complete the solution of the forces acting on two equal solid spheres moving in an unbounded medium along their line of centers. For unequal spheres, Davis (1969) has calculated the hydrodynamic forces when one sphere is in motion while the other is at rest. Bart (1968) solved the problem of a fluid drop settling toward a flat fluid interface.

Approximate solutions of the motion of two spheres using the method of reflection, are described by Happel & Brenner (1965). Hetsroni & Haber (1971) solved the motion of two droplets using this method, and computed the trajectories and efficiencies of their collisions. However, the convergence of the method of reflection may be poor when the two droplets are close (and that is when the collision does occur). Most recently, Wacholder & Weihs (1972) used the bispherical coordinate system to solve the motion of two spherical droplets falling along their line of center. They also computed numerically the correction to the Hadamard–Rybczynski drag force.

In this work we set out to find the general solution for the motion of two liquid droplets moving along their line of centers. Our solution will be more general than previous ones (Wacholder & Weihs 1972) since it yields the velocity fields for two drops which are unequal in size or viscosity. This will be a first part of a more general solution of the arbitrary motion of two liquid droplets in an unbounded quiescent incompressible fluid.

2. STATEMENT OF THE PROBLEM

The problem considered herein is that of two liquid droplets moving along their line of centers with low constant velocities $\mathbf{V}_a$ and $\mathbf{V}_b$, respectively, in an unbounded quiescent fluid. The droplets are spherical and their radii are a and b, respectively, as shown in figure 1.

The flow fields are assumed to be Stokesian and isothermal. The fluids involved are homogeneous, incompressible, Newtonian and have constant physical properties.

Thus, the governing field equations are:

for the interior of droplet a
$$\mu_a \nabla^2 \mathbf{U}_a = \nabla p_a \qquad [1a]$$
$$\nabla \cdot \mathbf{U}_a = 0 \qquad [1b]$$

for the interior of droplet b
$$\mu_b \nabla^2 \mathbf{U}_b = \nabla p_b \qquad [2a]$$
$$\nabla \cdot \mathbf{U}_b = 0 \qquad [2b]$$

for the field exterior to the droplets
$$\mu_e \nabla^2 \mathbf{u} = \nabla p \qquad [3a]$$
$$\nabla \cdot \mathbf{u} = 0 \qquad [3b]$$

where $\mathbf{U}_a$, p_a and $\mathbf{U}_b$, p_b are the velocity and the pressure fields, μ_a and μ_b are the viscosities

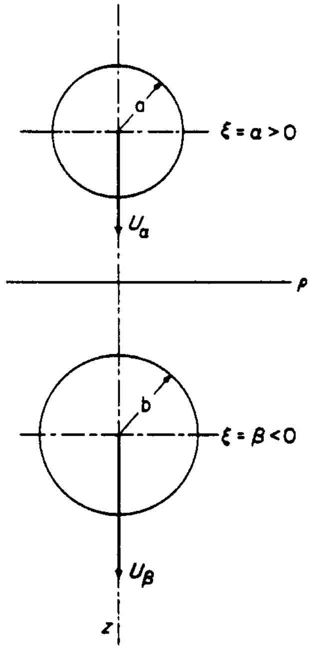

Figure 1. The bipolar coordinate system used in the analysis

of the droplets a and b, respectively; $\mathbf{u}$, p and μ_e are the velocity, pressure and viscosity of the field exterior to the droplets.

The boundary conditions are based on the following assumptions:

(a) The tangential components of the velocity vectors inside and outside of the droplets are continuous on the interface.
(b) The mass flow through the interface of the droplets vanishes.
(c) The tangential components of the normal stress vectors of the fluids interior and exterior to the droplets are continuous through the interface.
(d) The normal component of the normal stress vectors have a discontinuity which is proportional to the surface tension σ. Presently we shall assume that the droplets are spherical, and will not use this boundary condition. This implies that the ratio $\mu_e U/\sigma \ll 1$. A complete discussion of the use of this boundary condition is given elsewhere (e.g. Hetsroni *et al.* 1970a, b).
(e) Far from the droplets the flow field is unperturbed and we can assume, without any loss of generality, that the velocity vector vanishes. Thus, the boundary conditions are:

at the interface of droplet a $$\mathbf{U}_a = \mathbf{u} \qquad [4a]$$

$$\mathbf{V}_a \cdot \mathbf{i}_{n(a)} = \mathbf{u} \cdot \mathbf{i}_{n(a)} \qquad [4b]$$

$$\mathbf{\Pi}_{(n)(a)} = \boldsymbol{\tau}_{(n)(a)} + \sigma_{(a)}\left[\frac{1}{R_{1(a)}} + \frac{1}{R_{2(a)}}\right]\mathbf{i}_{n(a)} \qquad [4c]$$

at the interface of droplet b

$$\mathbf{U}_b = \mathbf{u} \quad [5a]$$

$$\mathbf{V}_b \cdot \mathbf{i}_{n(b)} = \mathbf{u} \cdot \mathbf{i}_{n(b)} \quad [5b]$$

$$\mathbf{\Pi}_{(n)(b)} = \boldsymbol{\tau}_{(n)(b)} + \sigma_{(b)} \left[\frac{1}{R_{1(b)}} + \frac{1}{R_{2(b)}} \right] \mathbf{i}_{n(b)} \quad [5c]$$

and far from the droplets

$$\mathbf{u} = 0 \quad [6]$$

where $\mathbf{i}_{n(a)}$, $\mathbf{i}_{n(b)}$ are unit vectors normal to the interface of droplet a and b, respectively; $\boldsymbol{\tau}_{(n)(a)}$ and $\boldsymbol{\tau}_{(n)(b)}$ are the normal stress vectors interior to droplets a and b; $\mathbf{\Pi}_{(n)}$ is the normal stress vector exterior to the droplets. $\sigma_{(a)}$ and $\sigma_{(b)}$ are the respective surface tensions and R_1 and R_2 are the principal radii.

3. THE SOLUTION

Due to bipolar geometry of the problem, the solution is best obtained in such coordinate system. The properties of the bipolar coordinate system are described by Whittaker & Watson (1920) (here we shall use the definitions of ρ and ξ as used by Stimson & Jeffery (1926). This differs from Whittaker & Watson (1920) by the fact that the symbols ρ and ξ are interchanged). Since the problem is axially symmetrical, the relationship between the bipolar coordinates and the cylindrical ones is given by:

$$z + i\rho = ic \cot \tfrac{1}{2}(\zeta + i\xi) \quad [7a]$$

where c is a positive constant; thus

$$\rho = c\frac{\sin \zeta}{\cosh \xi - \cos \zeta} \quad z = c\frac{\sinh \xi}{\cosh \xi - \cos \zeta}. \quad [7b]$$

The interface of the two droplets will be defined by $\xi = \alpha > 0$ and $\xi = \beta < 0$. The radii of the droplets are given by

$$a = c \operatorname{cosech} \alpha \quad \text{and} \quad b = -c \operatorname{cosech} \beta \quad [8a]$$

while the distance between the centers of the droplets is

$$l = c(\coth \alpha - \coth \beta). \quad [8b]$$

The stream function is defined in the usual way, namely:

$$u_\xi = \frac{h}{\rho}\frac{\partial \psi}{\partial \zeta} \quad u_\zeta = -\frac{h}{\rho}\frac{\partial \psi}{\partial \xi} \quad [9]$$

where

$$h = \frac{1}{c}(\cosh \xi - \cos \zeta). \quad [10]$$

The equation for the steady state creeping fluid motion is

$$E^4\psi = 0,$$

the general solution of which was given by Stimson & Jeffery (1926) as:

$$\psi = (\cosh \xi - \cos \zeta)^{-3/2} \sum_{n=1}^{\infty} W_n(\xi) C_{n+1}^{-1/2}(\cos \zeta) \qquad [11]$$

where $W_n(\xi) = a_n \cosh(n - \frac{1}{2})\xi + b_n \sinh(n - \frac{1}{2})\xi + c_n \cosh(n + \frac{3}{2})\xi + d_n \sinh(n + \frac{3}{2})\xi$ [12]

and where $C_{n+1}^{-1/2}(\cos \zeta) = C_{n+1}^{-1/2}(\mu)$ is the Gegenbauer polynomial of order $(n + 1)$ and degree $-1/2$. These functions are related to the Legendre polynomials via the relation

$$C_n^{-1/2}(\mu) = \frac{P_{n-2}(\mu) - P_n(\mu)}{2n - 1}. \qquad [13]$$

The constants a_n, b_n, c_n and d_n are determined from the boundary conditions. The boundary conditions, rewritten in the bipolar coordinate system, are at the interface of droplet a, i.e. at $\xi = \alpha$:

$$u_\xi = \mathbf{V}_\xi \cdot \mathbf{i}_\xi \qquad [14]$$

$$u_\xi = U_{\alpha\xi} \qquad [15a]$$

$$u_\zeta = U_{\alpha\zeta} \qquad [15b]$$

$$\Pi_{\xi\zeta} = \tau_{\alpha\xi\zeta} \qquad [15c]$$

at the interface of droplet b, i.e. at $\xi = \beta$

$$u_\xi = \mathbf{V}_\beta \cdot \mathbf{i}_\xi \qquad [16]$$

$$u_\xi = U_{\beta\xi} \qquad [17a]$$

$$u_\zeta = U_{\beta\zeta} \qquad [17b]$$

$$\Pi_{\xi\zeta} = \tau_{\beta\xi\zeta}. \qquad [17c]$$

Recall that the points $\xi = \pm\infty$ are inside the droplets. Therefore, in order to maintain the finiteness of the solution at the centers of the droplets, we must have for droplet a

$$a_n = -b_n \quad \text{and} \quad c_n = -d_n \qquad [18]$$

and for droplet b

$$a_n = b_n \quad \text{and} \quad c_n = d_n. \qquad [19]$$

The stream functions interior to droplets a and b and exterior to them thus simplify to:

$$\psi_\alpha = (\cosh \xi - \cos \zeta)^{-3/2} \sum_{n=1}^{\infty} W_n^\alpha(\xi) C_{n+1}^{-1/2}(\mu) \qquad [20]$$

$$\psi_\beta = (\cosh \xi - \cos \zeta)^{-3/2} \sum_{n=1}^{\infty} W_n^\beta(\xi) C_{n+1}^{-1/2}(\mu) \qquad [21]$$

$$\psi = (\cosh \xi - \cos \zeta)^{-3/2} \sum_{n=1}^{\infty} W_n(\xi) C_{n+1}^{-1/2}(\mu) \qquad [22]$$

where
$$W_n^{\alpha}(\xi) = A_n^{\alpha} \exp[-(n - \tfrac{1}{2})\xi] + C_n^{\alpha} \exp[-(n + \tfrac{3}{2})\xi]; \quad \xi > 0 \qquad [23]$$

$$W_n^{\beta}(\xi) = A_n^{\beta} \exp[(n - \tfrac{1}{2})\xi] + C_n^{\beta} \exp[(n + \tfrac{3}{2})\xi]; \quad \xi < 0 \qquad [24]$$

and where $W_n(\xi)$ is defined by [12], which satisfies automatically the boundary conditions far from the droplet, i.e. [6].

In order to proceed with the solution and determine the coefficients A_n^{α}, C_n^{α}, A_n^{β}, etc., the boundary conditions are expressed in terms of the stream function (Appendix A).

Now we substitute [20], [21] & [22] into the boundary conditions, [A1]–[A4] and obtain:

For droplet a, i.e. at $\xi = \alpha$

$$\sum_{n=1}^{\infty} W_n(\alpha) C_{n+1}^{-1/2}(\mu) = \sum_{n=1}^{\infty} W_n^{\alpha}(\alpha) C_{n+1}^{-1/2}(\mu) \qquad [25]$$

$$\sum_{n=1}^{\infty} \frac{\mathrm{d}W_n(\alpha)}{\mathrm{d}\xi} C_{n+1}^{-1/2}(\mu) = \sum_{n=1}^{\infty} \frac{\mathrm{d}W_n^{\alpha}(\alpha)}{\mathrm{d}\xi} C_{n+1}^{-1/2}(\mu) \qquad [26]$$

$$\sum_{n=1}^{\infty} W_n^{\alpha}(\alpha) C_{n+1}^{-1/2}(\mu) = -\frac{\sin^2 \zeta}{2(\cosh \alpha - \cos \zeta)^{1/2}} V_{\alpha} C^2 \qquad [27]$$

$$\sum_{n=1}^{\infty} \left(\frac{\mathrm{d}^2 W_n(\alpha)}{\mathrm{d}\xi^2} - \lambda_a \frac{\mathrm{d}^2 W_n^{\alpha}(\alpha)}{\mathrm{d}\xi^2} \right) C_{n+1}^{-1/2}(\mu)$$

$$= (1 - \lambda_a) V_a C^2 \left[\frac{\cosh \alpha \sin^2 \zeta}{4(\cosh \alpha - \cos \zeta)^{1/2}} - \frac{3 \sin^2 \zeta \sinh^2 \alpha}{8(\cosh \alpha - \cos \zeta)^{5/2}} \right] \qquad [28]$$

where $\lambda_a = \mu_a/\mu_e$ is the ratio between the viscosity of droplet a to that of the external field.

For droplet b, i.e. at $\xi = \beta$, one obtains an identical set of equations, with β replacing α everywhere.

To continue the solution we express the right-hand side of [27] & [28] (and the equivalent equations for $\xi = \beta$), as an infinite sum of Gegenbauer polynomials $C_{n+1}^{-1/2}(\mu)$ (Appendix B). Then, using the orthogonality properties of the Gegenbauer polynomials, eight equations are obtained for the eight unknown coefficients a_n, b_n, c_n, d_n (of the continuous field) A_n^{α}, C_n^{α}, A_n^{β} and C_n^{β} (of the droplets). Once these coefficients are obtained, the solution of the flow fields is completed and the stream functions [11] & [12] are known, and it is not necessary to write down the details. However, in order to compute the terminal settling velocities of the droplets, it is necessary to compute in detail the drag force acting on each droplet.

4. THE DRAG FORCE

It was shown quite generally by Stimson & Jeffery (1926) that the drag force of droplets (spheres) a and b are given by:

$$F_a = \frac{2\sqrt{(2)}\pi\mu_e}{c} \sum_{n=1}^{\infty} (a_n + b_n + c_n + d_n) \qquad [29]$$

$$F_b = -\frac{2\sqrt{(2)}\pi\mu_e}{c}\sum_{n=1}^{\infty}(a_n - b_n + c_n - d_n). \qquad [30]$$

After rather tedious algebra the sums of the coefficients in [29] were computed. The drag force on droplet a is thus given by:

$$F_a = -6\pi\mu_e a[\Lambda_{\alpha\alpha}V_a + \Lambda_{\alpha\beta}V_b] \qquad [31]$$

where

$$\Lambda_{\alpha\alpha} = \frac{\sqrt{2}\sinh\alpha}{3c}\sum_{n=1}^{\infty}\frac{\delta_o + \lambda_a\delta_1 + \lambda_b\delta_2 + \lambda_a\lambda_b\delta_3}{\Delta} \qquad [32]$$

$$\Lambda_{\alpha\beta} = \frac{\sqrt{2}\sinh\alpha}{3c}\sum_{n=1}^{\infty}\frac{\bar{\delta}_o + \lambda_a\bar{\delta}_1 + \lambda_b\bar{\delta}_2 + \lambda_a\lambda_b\bar{\delta}_3}{\Delta}. \qquad [33]$$

The coefficients δ_o, δ_1, etc. are defined in Appendix C. The two coefficients $\Lambda_{\alpha\alpha}$ and $\Lambda_{\alpha\beta}$ were evaluated numerically and the results are depicted in figures 2–5.

In figure 2 the values of $+\Lambda_{\alpha\alpha}$ and $-\Lambda_{\alpha\beta}$ are shown versus the ratio between the radii of the droplets $p = a/b$, for the case when their viscosities are equal and very small compared to that of the infinite medium (e.g. air bubbles in viscous liquid), and for the case when the ratio between the radius of one droplet to the distance between them is rather small, i.e. $a/l = 0.05$. Notice that $\Lambda_{\alpha\alpha} \to 2/3$ as $p \to 1$, as is indeed expected.

In figure 3 the values of $+\Lambda_{\alpha\alpha}$ and $-\Lambda_{\alpha\beta}$ are depicted versus the ratio between the radii, for the case when the viscosities of the droplets are equal to that of the surrounding fluid. Here the distance between the droplets was taken to be $10a$.

In figure 4 the values of $+\Lambda_{\alpha\alpha}$ and $-\Lambda_{\alpha\beta}$ are plotted versus the ratio of the viscosity of the droplets (assumed equal, i.e. $\lambda_a = \lambda_b = \lambda$) to that of the infinite medium. Here the droplets were taken to be of the same size, i.e. $a = b$ and the distance between them is a parameter.

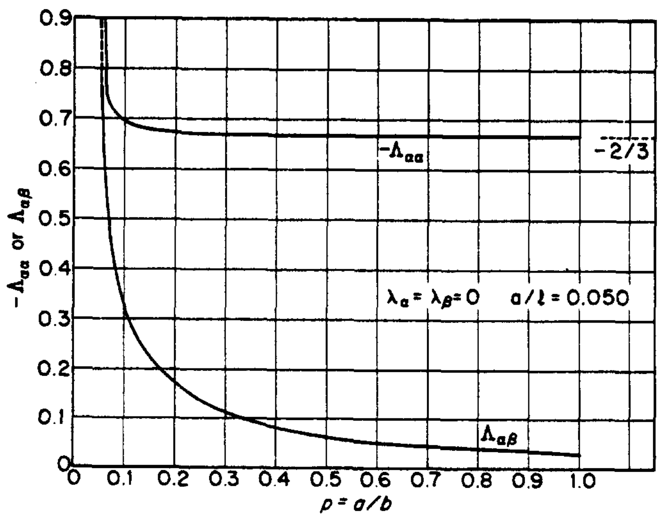

Figure 2. The correction factors $-\Lambda_{\alpha\alpha}$ and $\Lambda_{\alpha\beta}$ in the equation for the drag force [31], versus the ratio between the radii p. The viscosities $\lambda_a = \lambda_b = 0$ and the radius/distance ratio $(a/l) = 0.05$.

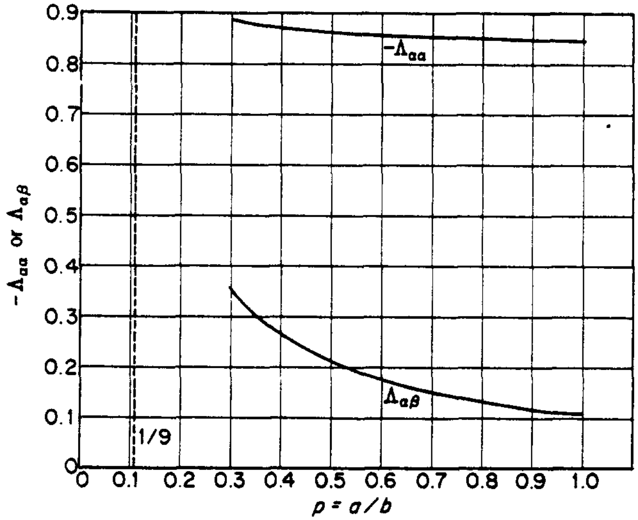

Figure 3. The correction factors $-\Lambda_{\alpha\alpha}$ and $\Lambda_{\alpha\beta}$ in the equation for the drag force [31], versus the ratio between the radii p. The viscosities of the droplets are equal to that of the medium $\lambda_a = \lambda_b = 1$ and the radius/distance ratio $(a/l) = 0.1$.

In figure 5 the correction factors $+\Lambda_{\alpha\alpha}$ and $-\Lambda_{\alpha\beta}$ are shown versus the ratio between the radii. Here the viscosity of the droplets was taken to be 67 times that of the surrounding liquid (as in the case of water droplets in the atmosphere). The distance between the droplets was 3.3 times the radius of one droplet. Notice that as $p \to \infty$, i.e. one droplet becomes very much larger compared to the second one, $+\Lambda_{\alpha\alpha} \to 1$ and $\Lambda_{\alpha\beta} \to 0$, as indeed is expected from other solutions.

In all the figures it can be observed that the correction coefficient $\Lambda_{\alpha\alpha}$ of one drop is less affected by the second drop than the correction coefficient $\Lambda_{\alpha\beta}$. Also, both correction coefficients increase without bound when the two droplets come closer to each other. When the two radii are equal, the velocities are equal and the sum of the two coefficients is bounded (as discussed further in Section 5). However, for unequal drops, the settling velocities are unequal (i.e. $V_a \neq V_b$) and the drag force increases without bound.

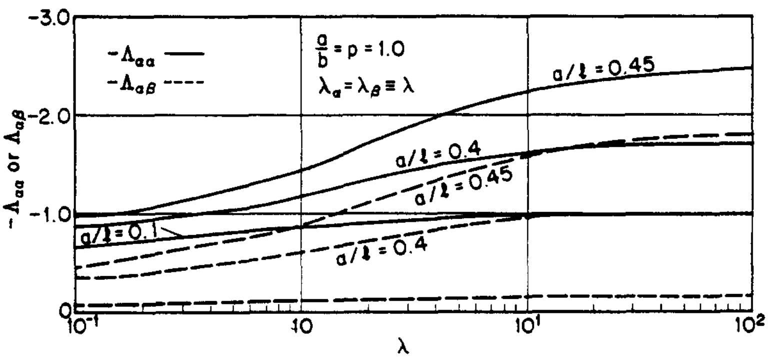

Figure 4. The correction factors $-\Lambda_{\alpha\alpha}$ and $\Lambda_{\alpha\beta}$ in the equation for the drag force [31], versus the viscosity ratio λ. The radii of the droplets are equal and the distance between them is a parameter.

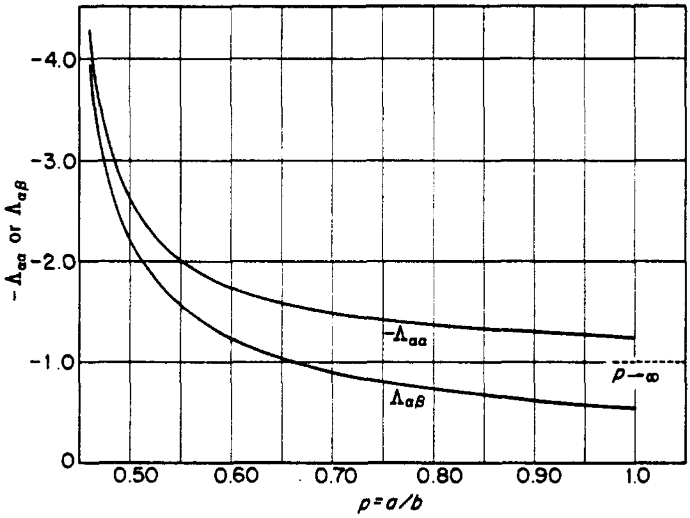

Figure 5. The correction factors $-\Lambda_{\alpha\alpha}$ and $\Lambda_{\alpha\beta}$ in the equation for the drag force [31], versus the ratio between the radii of the droplets p. The viscosities are equal $\lambda_a = \lambda_b = 67$ and the radius/distance ratio $(a/l) = 0.3$.

Notice that the drag force computed herein and given in [31] is the most general one, and includes all previous solutions as special cases. Thus, when $\beta = 0$ (i.e. $b = \infty$) our solution reduces to that of Bart (1968). When the droplets are equal, i.e. $a = b$ and $\lambda_a = \lambda_b$, our solution simplifies to that of Wacholder & Weihs (1972) (there is some numerical discrepancy, which we believe is caused by an error in their calculations); when $\lambda_a = \lambda_b = \infty$ and $a = b$, the results of Stimson & Jeffery (1926) and Goldman *et al.* (1967) are obtained; when one droplet vanishes, i.e. $b = o$ or $l/b \to \infty$ [31] simplifies to the celebrated Hadamard–Rybczynski solution.

One limiting case of particular interest is discussed in the next Section.

5. TWO EQUAL DROPLETS IN CONTACT

For two equal size droplets the coefficients b_n and d_n in [29] and [30] vanish and the forces are equal. The force can then be conveniently written as

$$F_D = -6\pi\mu_e aV\Lambda' \qquad [34]$$

where a is the radius of any one of the droplets and Λ' is a correction factor. This correction factor was computed by Faxén (who corrected a numerical error in Stimson & Jeffery's work) for two solid spheres. Faxén (1927) also computed Λ' for the limiting case when the two spheres touch.

For two equal-size droplets in contact but *not* of same viscosity we express the drag force in the general form

$$F_D = -6\pi\mu_e aV_a \frac{2/3 + \lambda_a}{1 + \lambda_a} \Lambda \qquad [35]$$

where the coefficient Λ is defined

$$\Lambda = \frac{1 + \lambda_a}{2/3 + \lambda_a} \cdot \frac{1}{12} \int_o^\infty \frac{C_o + \lambda_a C_1 + \lambda_b C_2 + \lambda_a \lambda_b C_3}{A_o + (\lambda_a + \lambda_b) A_1 + \lambda_a \lambda_b A_2} \mathrm{d}x \qquad [36]$$

where

$$C_0 = 64e^{-x} \sinh^2 x(\cosh x + xe^x) \qquad [37a]$$

$$C_1 = 8e^{2x}(2x^2 + 2x + 1) - 8e^{-2x}(2x - 1) - 8e^{-4x} - 48x^2 - 32x - 8 \qquad [37b]$$

$$C_2 = 8e^{-2x}(2x^2 - 2x + 1) + 8e^{2x}(2x + 1) - 8e^{-4x} - 48x^2 - 32x - 8 \qquad [37c]$$

$$C_3 = (16 + 32x^2)\sinh 2x + 32x \cosh 2x + 8e^{-4x} - 64x^3 - 64x^2 - 32x - 8 \qquad [37d]$$

$$A_0 = 4 \sinh^2 2x \qquad [37e]$$

$$A_1 = 2 \sinh 4x - 8x \qquad [37f]$$

$$A_2 = 4 \sinh^2 2x - 16x^2. \qquad [37g]$$

The coefficients C_0, C_1, C_2, C_3, A_0, A_1 and A_2 were obtained by using the equations in Appendix C, with $\alpha \to 0$, $\beta \to 0$, $n\alpha \equiv x$. Expressions of the order $n^p\alpha^q$ with $p < q$ were neglected and the summation was replaced by integration at the limit $\alpha \to 0$. The coefficients Λ for various values of λ_a are given in table 1, for the case when $\lambda_a = \lambda_b$. The computations in the table were done to accuracy $O(10^{-5})$.

Table 1. Correction factor for the drag force in [35].

λ	Λ	λ	Λ
0.0	0.69309765	5.0	0.65323899
0.5	0.67778645	7.0	0.65119544
1.0	0.66967093	9.0	0.64997234
1.5	0.66474567	15.0	0.64814263
2.0	0.66145397	30.0	0.64667191
2.5	0.65910276	50.0	0.64605811
3.0	0.65734077	100.0	0.64558767
		1000.0	0.64514946

Notice that as $\lambda \to \infty$ (i.e. the droplets approach solid spheres), our solution converges rapidly to the value of 0.645 given by Faxén. Notice also that since the variation in Λ is rather small (in the order of 6 per cent), the change in Λ' in [34] is dominated by the term $[(2/3 + \lambda)/(1 + \lambda)]$.

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REFERENCES

BART, E. 1968 The slow unsteady settling of a fluid sphere toward a flat fluid interface. *Chem. Engng Sci.* **23**, 193–210.

COOLEY, M. D. A. & O'NEILL, M. E. 1969 On the slow motion generated in a viscous fluid by the approach of a sphere to a plane wall or stationary sphere. *Mathematika* **16**, 37–49.

COX, R. G. & BRENNER, H. 1967 The slow motion of a sphere through a viscous fluid towards a plane surface—II. Small gap widths, including inertial effects. *Chem. Engng Sci.* **22**, 1753–1777.

DAVIS, M. H. 1969 The slow translation and rotation of two unequal spheres in a viscous fluid. *Chem. Engng Sci.* **24**, 1769–1776.

DAVIS, M. H. & SARTOR, J. D. 1967 Theoretical collision efficiencies for small cloud droplets in Stokes flow. *Nature* **215**, 1371–1372.

DEAN, W. R. & O'NEILL, M. E. 1963 The slow motion of viscous liquid caused by the rotation of a solid sphere. *Mathematika* **10**, 13–27.

FAXÉN, H. 1927 Die Geschwindigkeit zwen Kugeln, die unter Ein wirkung der Schwere in einen zahen Flussigkeit fallen. *ZAMM* **7**, 79–80.

GOLDMAN, A. J., COX, R. G. & BRENNER, H. 1967 Slow viscous motion of a sphere parallel to a plane wall, Part I. *Chem. Engng Sci.* **22**, 637; Part II. *ibid.* **22**, 653–660.

GREENSTEIN, T. 1972 Slow settling of two droplets symmetrically placed about the axis of a circular cylinder in a direction perpendicular to their line of centers. *J. Phy. Soc. Japan* **32**, 1398–1402.

HABER, S. & HETSRONI, G. 1971 The dynamics of a deformable drop suspended in an unbounded Stokes flow. *J. Fluid Mech.* **49**, 257–277.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

HETSRONI, G. & HABER, S. 1970a The flow field in and around a droplet or bubble submerged in an unbounded arbitrary velocity field. *Rheo. Acta.* **9**, 488–496.

HETSRONI, G., HABER, S. & WACHOLDER, E. 1970b The flow field in and around a droplet moving axially within a tube. *J. Fluid Mech.* **41**, 689.

HETSRONI, C. & HABER, S. 1971 Low Reynolds number motion of raindrops in the atmosphere. Dep. of Mechanical Engineering, Technion, Haifa, Israel.

HOCKING, L. M. 1959 The collision efficiency of small drops. *Q. J. Roy. Met. Soc.* **85**, 44–50.

HOCKING, E. M. & JONAS, P. R. 1970 The collision efficiency of small drops, *Q. J. Roy. Met. Soc.* **96**, 722–729.

JEFFERY, G. B. 1915 On the steady rotation of a solid of revolution in a viscous fluid. *Proc. London Math. Soc.* (Ser. 2) **14**, 327.

O'NEILL, M. E. 1970 Exact solutions of the equations of slow viscous flow generated by the asymmetrical motion of two equal spheres. *Appl. Sci. Res.* **21**, 453.

STIMSON, M. & JEFFERY, G. B. 1926 The motion of two spheres in viscous fluid. *Proc. Roy. Soc.* **A111**, 110.

WACHOLDER, E. & WEIHS, D. 1972 Slow motion of a fluid sphere in the vicinity of another sphere or a plane boundary. *Chem. Engng Sci.* **27**, 1817–1828.

WHITTAKER, E. T. & WATSON, G. N. 1920 *Modern Analysis*, Cambridge University Press.

Sommaire—Des solutions exactes sont obtenues pour le régime d'écoulement fluant quasi permanent à l'intérieur et à l'extérieur de deux gouttelettes sphériques en mouvement le long de leur ligue des centres. Des résultats numériques sont présentés. Ils constituent une généralisation des toutes les solutions précédents.

Auszug—Es werden genaue Gleichungen für den innen und außen an zwei kugelförmigen Tröpfchen quasi im Beharrungszustand bestehenden Kriechfluß abgeleitet, welche sich entlang ihrer Linie der Mittelpunkte bewegen. Es werden zahlenmäßige Ergebnisse gebracht, welche alle früheren Lösungen als Spezialfälle einschließen.

Резюме—Находят точные решения для квазистационарной установившейся ползучести с внутренней и наружной сторон двух круглых капелек двигающихся вдоль своих линий центров. Представляют численные результаты, включающие как специальные случаи все прошлые решения.

APPENDIX A

Boundary conditions

Expressing the boundary conditions, [15a, b] in terms of the stream function defined by [9], one obtains homogeneous equations in ψ, at the interface of droplet a, i.e. at $\xi = \alpha$:

$$\psi = \psi_\alpha \qquad \text{[A-1]}$$

$$\partial\psi/\partial\xi = \partial\psi_\alpha/\partial\xi. \qquad \text{[A-2]}$$

Expressing [14] in terms of the stream function, at $\xi = \alpha$:

$$\frac{h}{\rho}\frac{\partial\psi_\alpha}{\partial\zeta} = V_\alpha(k\cdot\mathbf{i}_t) = V_\alpha(\mathbf{i}_\zeta\cdot\nabla z) = V_\alpha\left(h\frac{\partial z}{\partial\xi}\right) = V_\alpha\frac{1-\cosh\alpha\cos\zeta}{\cosh\alpha-\cos\zeta}$$

for which
$$\frac{\partial\psi_\alpha}{\partial\zeta} = V_\alpha c^2\left[\frac{\cosh\alpha\sin\zeta}{(\cosh\alpha-\cos\zeta)^2} - \frac{\sinh\alpha\sin\zeta}{(\cosh\alpha-\cos\zeta)^3}\right]$$

by simple integration
$$\psi_\alpha = -V_\alpha c^2\left[-\frac{\cosh\alpha}{(\cosh\alpha-\cos\zeta)} + \frac{\sinh^2\alpha}{2(\cosh\alpha-\cos\zeta)^2} + d\right]$$

where d is a constant of integration. Here we choose $d = \frac{1}{2}$, without any loss of generality, and get

$$\psi_\alpha|_{\xi=\alpha} = -V_\alpha c^2\left[\frac{1-\cos^2\zeta}{2(\cosh\alpha-\cos\zeta)^2}\right] \qquad \text{[A-3]}$$

An identical equation is obtained for the stream function interior to droplet b, i.e. ψ_β at the interface $\xi = \beta$, when β replaces α everywhere in [A-3].

To express [15c] & [17c] in terms of the stream function, first write the stress in terms of the velocity:

$$\Pi_{\xi\zeta} = \mathbf{i}_\xi\cdot\mathbf{\Pi}\cdot\mathbf{i}_\zeta = \mathbf{i}_\xi\cdot[-p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)]\mathbf{i}_\zeta = \mu h\left[u_{\xi,\zeta} - hu_\zeta\left(\frac{1}{h}\right)_{,\xi} + u_{\zeta,\xi} - hu_\xi\left(\frac{1}{h}\right)_{,\zeta}\right].$$

Finally
$$\Pi_{\xi\zeta} = \mu h(u_{\xi,\zeta} + u_{\zeta,\xi}) + \frac{\mu}{c}(u_\zeta\sinh\xi + u_\xi\sin\zeta).$$

A similar expression is obtained for $\tau_{\alpha\xi\zeta}$ (i.e. the stress interior to droplet a), when U_α replaces u and μ_α replaces μ, viz.

$$\tau_{\alpha\xi\zeta} = \mu_\alpha h(U_{\alpha\xi,\zeta} + U_{\alpha\zeta,\xi}) + \frac{\mu_\alpha}{c}(U_{\alpha\zeta}\sinh\xi + U_{\alpha\xi}\sin\zeta).$$

An identical expression is obtained for $\tau_{\beta\xi\zeta}$, when β replaces α everywhere. The comma between the subscripts denotes differentiation.

Substituting the last two equations and [9] into [15c], and making use of [A-1] through [A-3], one obtains at $\xi = \alpha$:

$$(1-\lambda_\alpha)\rho\left[\frac{\partial}{\partial\zeta}\left(\frac{h}{\rho}\right)\frac{\partial\psi_\alpha}{\partial\zeta} + \frac{h}{\rho}\frac{\partial^2\psi_\alpha}{\partial\zeta^2}\right] + \frac{1}{c}(1-\lambda_\alpha)\left[-\frac{\partial\psi_\alpha}{\partial\xi}\sinh a + \frac{\partial\psi_\alpha}{\partial\zeta}\sin\zeta\right]$$
$$+ (1-\lambda_\alpha)\rho\left[\frac{\partial}{\partial\xi}\left(\frac{h}{\rho}\right)\frac{\partial\psi_\alpha}{o\xi}\right] - h\left(\frac{\partial^2\psi}{\partial\xi^2} - \lambda_\alpha\frac{\partial^2\psi_\alpha}{\partial\xi^2}\right) = 0.$$

Substitution of [7b], [20], [22] & [A-3] into the above equation yields, after somewhat lengthy algebra, at $\xi = \alpha$:

$$\sum_{n=1}^{\infty}\left(\frac{d^2W_n(\alpha)}{d\xi^2} - \lambda_\alpha\frac{d^2W^\alpha_{n(\alpha)}}{d\xi^2}\right)C^{-1/2}_{n+1}(\mu)$$
$$= (1-\lambda_\alpha)V_\alpha C^2\left[\frac{\cosh\alpha\sin^2\zeta}{4(\cosh\alpha-\cos\zeta)^{1/2}} - \frac{3}{8}\frac{\sin^2\zeta\sinh^2\alpha}{(\cosh\alpha-\cos\zeta)^{5/2}}\right]. \qquad \text{[A-4]}$$

Similar equation is obtained at $\xi = \beta$, when β replaces α everywhere in [A-4].

APPENDIX B

The right-hand side of [27] & [28] is to be expressed as in infinite sum of Gegenbauer polynomials $C^{-1/2}_{n+1}$. First we avail ourselves of the general expression (Whittaker & Watson 1920):

$$(1-2h\mu+h^2)^{-\nu} = \sum_{n=0}^{\infty} h^n c^\nu_n(\mu). \qquad \text{[B-1]}$$

Differentiating and putting $\nu = +\frac{1}{2}$:

$$\frac{(1-\mu^2)^2}{(1-2h\mu+h^2)^{3/2}} = \sum_{n=0}^{\infty} h^n(n+1)(n+2)C^{-1/2}_{n+2}(\mu) \qquad \text{[B-2]}$$

and
$$\frac{(1-\mu^2)}{(1-2h\mu+h^2)^{5/2}} = \frac{1}{3}\sum_{n=0}^{\infty} h\frac{n-2(n-1)n(n+1)(n+2)}{(2n+1)}(C^{-1/2}_n - C^{-1/2}_{n+2}). \qquad \text{[B-3]}$$

Setting $h = e^{-\alpha}$, [27] & [28] can be expressed as infinite sums of Gegenbauer polynomials $C^{-1/2}_{n+1}$. Identical procedure is followed for the second droplet, where β replaces α everywhere. Using the orthogonality properties of these polynomials, eight equations for the eight unknown coefficients (viz. a_n, b_n, c_n, d_n, A^α_n, C^α_n, A^β_n, C^β_n) are obtained as follows.

$$a_n \cosh(n - \tfrac{1}{2})\alpha + b_n \sinh(n - \tfrac{1}{2})\alpha + c_n \cosh(n + \tfrac{3}{2})\alpha + d_n \sinh(n + \tfrac{3}{2})\alpha = K_1 f_n(\alpha) \quad \text{[B-4]}$$

$$a_n \cosh(n - \tfrac{1}{2})\beta + b_n \sinh(n - \tfrac{1}{2})\beta + c_n \cosh(n + \tfrac{3}{2})\beta + d_n \sinh(n + \tfrac{3}{2})\beta = K_1 f_n(-\beta) \quad \text{[B-5]}$$

$$A_n^\alpha e^{-(n-1/2)\alpha} + C_n^\alpha e^{-(n+3/2)\alpha} = K_1 f_n(\alpha) \quad \text{[B-6]}$$

$$A_n^\beta e^{(n-1/2)\beta} + C_n^\beta e^{(n+3/2)\beta} = K_1 f_n(-\beta) \quad \text{[B-7]}$$

$$(n - \tfrac{1}{2})[a_n \sinh(n - \tfrac{1}{2})\alpha + b_n \cosh(n - \tfrac{1}{2})\alpha + A_n^\alpha e^{-(n-1/2)\alpha}] + (n + \tfrac{3}{2})[c_n \sinh(n + \tfrac{3}{2})\alpha + d_n \cosh(n + \tfrac{3}{2})\alpha - C_n^\alpha e^{(n+3/2)\alpha}] = 0 \quad \text{[B-8]}$$

$$(n - \tfrac{1}{2})[a_n \sinh(n - \tfrac{1}{2})\beta + b_n \cosh(n - \tfrac{1}{2})\beta - A_n^\beta e^{(n-1/2)\beta}] + (n + \tfrac{3}{2})[c_n \sinh(n + \tfrac{3}{2})\beta + d_n \cosh(n + \tfrac{3}{2})\beta - C_n^\beta e^{(n+3/2)\beta}] = 0 \quad \text{[B-9]}$$

$$(n - \tfrac{1}{2})^2[a_n \cosh(n - \tfrac{1}{2})\alpha + b_n \sinh(n - \tfrac{1}{2})\alpha - \lambda_a A_n^\alpha e^{-(n-1/2)\alpha}] + (n + \tfrac{3}{2})^2[c_n \cosh(n + \tfrac{3}{2})\alpha + d_n \sinh(n + \tfrac{3}{2})\alpha - \lambda_a C_n^\alpha e^{-(n+3/2)\alpha}] = (1 - \lambda_a)K_2 g_n(\alpha) \quad \text{[B-10]}$$

$$(n - \tfrac{1}{2})^2[a_n \cosh(n - \tfrac{1}{2})\beta + b_n \sinh(n - \tfrac{1}{2})\beta - \lambda_b A_n^\beta e^{(n-1/2)\beta}] + (n + \tfrac{3}{2})^2[c_n \cosh(n + \tfrac{3}{2})\beta + d_n \sinh(n + \tfrac{3}{2})\beta - \lambda_b C_n^\beta e^{(n+3/2)\beta}] = (1 - \lambda_b)K_2 g_n(-\beta) \quad \text{[B-11]}$$

where

$$K_1 = \frac{c^2}{\sqrt{2}} \frac{n(n+1)}{(2n-1)(2n+3)} \quad \text{[B-12]}$$

$$K_2 = \frac{c^2}{4\sqrt{2}} n(n+1) \quad \text{[B-13]}$$

$$f_n(\alpha) = U_\alpha[(2n - 1)e^{-(n+3/2)\alpha} - (2n + 3)e^{-(n-1/2)\alpha}] \quad \text{[B-14]}$$

$$g_n(\alpha) = U_\alpha[(2n + 3)e^{-(n+3/2)\alpha} - (2n - 1)e^{-(n-1/2)\alpha}] \quad \text{[B-15]}$$

and similar functions $f_n(-\beta)$ and $g_n(-\beta)$, with β replacing α everywhere in [B-14] and [B-15].

APPENDIX C

The coefficients in [31] are defined as follows:

$$\delta_o = 4(2n + 1)^2 K_1[(2n + 3)\sinh(n + \tfrac{3}{2})(\alpha - \beta)e^{-(n-1/2)(\alpha+\beta)} - (2n - 1)e^{-(n+3/2)(\alpha+\beta)} \sinh(n - \tfrac{1}{2})(\alpha - \beta)] \quad \text{[C-1]}$$

$$\bar{\delta}_o = 4(2n + 1)^2 K_1[-(2n + 3)e^{-(n-1/2)(\alpha-\beta)} \sinh(n + \tfrac{3}{2})(\alpha - \beta) + (2n - 1)e^{-(n+3/2)(\alpha-\beta)} \sinh(n - \tfrac{1}{2})(\alpha - \beta)] \quad \text{[C-2]}$$

$$\delta_1 = -(2n + 1)^2 K_1[2(2n - 1)(2n + 3)e^{-(2n+1)\beta} - (2n + 3)^2 e^{-(n-1/2)(\alpha+\beta)} \cosh(n + \tfrac{3}{2})(\alpha - \beta) - (2n - 1)^2 e^{-(n+3/2)(\alpha+\beta)} \cosh(n - \tfrac{1}{2})(\alpha - \beta) - (2n - 1)(2n + 3)e^{-(n-1/2)(\alpha+\beta)} \sinh(n + \tfrac{3}{2})(\alpha - \beta) - (2n - 1)(2n + 3)e^{-(n+3/2)(\alpha+\beta)} \sinh(n - \tfrac{1}{2}(\alpha - \beta)] \quad \text{[C-3]}$$

$$\bar{\delta}_1 = -(2n+1)^2 K_1[-2(2n-1)(2n+3)\cosh(2\beta) + (2n+3)^2 e^{-(n-1/2)(\alpha-\beta)} \cosh(n+\tfrac{3}{2})(\alpha-\beta)$$

$$+(2n-1)^2e^{-(n+3/2)(\alpha-\beta)}\cosh(n-\tfrac{1}{2})(\alpha-\beta)+(2n-1)(2n+3)e^{-(n-1/2)(\alpha-\beta)}\sinh(n+\tfrac{3}{2})(\alpha-\beta)$$
$$+(2n-1)(2n+3)e^{-(n+3/2)(\alpha-\beta)}\sinh(n-\tfrac{1}{2})(\alpha-\beta)] \qquad \text{[C-4]}$$

$$\delta_2 = -(2n+1)^2K_1[2(2n-1)(2n+3)e^{-(2n+1)\alpha}-(2n+3)^2e^{-(n-1/2)(\alpha+\beta)}\cosh(n+\tfrac{3}{2})(\alpha-\beta)$$
$$-(2n-1)^2e^{-(n+3/2)(\alpha+\beta)}\cosh(n-\tfrac{1}{2})(\alpha-\beta)+(2n-1)(2n+3)e^{-(n-1/2)(\alpha+\beta)}\sinh(n+\tfrac{3}{2})(\alpha-\beta)$$
$$+(2n-1)(2n+3)e^{-(n+3/2)(\alpha+\beta)}\sinh(n-\tfrac{1}{2})(\alpha-\beta)] \qquad \text{[C-5]}$$

$$\bar{\delta}_2 = -(2n+1)^2K_1[-2(2n-1)(2n+3)\cosh(2\alpha)+(2n+3)^2e^{-(n-1/2)(\alpha-\beta)}\cosh(n+\tfrac{3}{2})(\alpha-\beta)$$
$$+(2n-1)^2e^{-(n+3/2)(\alpha-\beta)}\cosh(n-\tfrac{1}{2})(\alpha-\beta)+(2n-1)(2n+3)e^{-(n-1/2)(\alpha-\beta)}\sinh(n+\tfrac{3}{2})(\alpha-\beta)$$
$$+(2n-1)(2n+3)e^{-(n+3/2)(\alpha-\beta)}\sinh(n-\tfrac{1}{2})(\alpha-\beta)] \qquad \text{[C-6]}$$

$$\delta_3 = 2(2n+1)^2K_1[(2n-1)(2n+3)(e^{-(2n+1)\beta}-e^{-(2n+1)\alpha})-(2n+1)(2n+3)e^{-(n-1/2)(\alpha+\beta)}$$
$$\sinh(n+\tfrac{3}{2})(\alpha-\beta)-(2n+1)(2n-1)e^{-(n+3/2)(\alpha+\beta)}\sinh(n-\tfrac{1}{2})(\alpha-\beta)] \qquad \text{[C-7]}$$

$$\bar{\delta}_3 = -(2n+1)^2K_1[-2(2n-1)(2n+1)(2n+3)\sinh(\alpha-\beta)\cosh(\alpha+\beta)+(2n+1)^3\sinh 2(\alpha-\beta)$$
$$+2(2n+1)^2\cosh 2(\alpha-\beta)-8e^{-(2n+1)(\alpha-\beta)}-2(2n-1)(2n+3)] \qquad \text{[C-8]}$$

$$\Delta = (2n+1)^2[4\sinh(n-\tfrac{1}{2})(\alpha-\beta)\sinh(n+\tfrac{3}{2})(\alpha-\beta)+(\lambda_a+\lambda_b)[2\sinh(2n+1)(\alpha-\beta)$$
$$-(2n+1)\sinh 2(\alpha-\beta)]+\lambda_a\lambda_b[4\sinh^2(n+\tfrac{1}{2})(\alpha-\beta)-(2n+1)^2\sinh^2(\alpha-\beta)] \qquad \text{[C-9]}$$